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THE FORCE ON AN ELASTIC SINGULARITY

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CONTENTS

The parallel between the classical theory of elasticity and the modern physical theory of the solid state is incomplete; the former has nothing analogous to the concept of the force acting on an imperfection (dislocation, foreign atom, etc.) in a stressed crystal lattice. To remedy this a general theory of the forces on singularities in a Hookean elastic continuum is developed. The singularity is taken to be any state of internal stress satisfying the equilibrium equations but not the compatibility conditions. The force on a singularity can be given as an integral over a surface enclosing it. The integral contains the elastic field quantities which would surround the singularity in an infinite medium, multiplied by the difference between these quantities and those actually present. The expression for the force is thus of essentially the same form whether the force is due to applied surface tractions, other singularities or the presence of the free surface of the body ('image force'). A region of inhomogeneity in the elastic constants modifies the stress field; if it is mobile one can define and calculate the force on it. The total force on the singularities and inhomogeneities inside a surface can be expressed in terms of the integral of a 'Maxwell tensor of elasticity' taken over the surface. Possible extensions to the dynamical case are discussed.

1. Introduction

Modern theories of the solid state make use of the idea of the forces acting on imperfections in a crystal lattice, such as, for example, dislocations, foreign atoms, vacant lattice points, grain boundaries. The stress in the material arises from the presence of the imperfections and from any externally applied surface and body forces. If the applied forces are held constant, the total energy of the system (internal energy of the body plus potential energy of the sources of external force) is a function of the set of parameters necessary to specify the configuration of the imperfections. The negative gradient of the total energy with respect to the position of an imperfection may conveniently be called the force on it. This force, in a sense fictitious, is introduced to give a picturesque description of energy changes, and must not be confused with the ordinary surface and body forces acting on the material.

Even if there is only a single imperfection in the body and no externally produced stresses the force on it will not in general vanish. This 'image force' will depend on the shape of the boundary of the body and on the variation of the elastic constants from point to point in it.

The extra term which appears when there are also stresses due to applied forces and to other imperfections can be regarded as the force which these exert on the original imperfection.

In the usual theory of the elastic continuum the analogue of an imperfection (or rather of the stress-field associated with it) is some state of internal stress not produced by surface or body forces, for example, a nucleus of strain (Love 1927). The elastic analogue of an interstitial atom is a centre of dilatation supplemented by point singularities of higher order. The dislocations of physical theory have, of course, their analogues in certain of the elastic dislocations of Volterra (1907).

In many calculations it is possible to replace an imperfection by its elastic counterpart, allowing for the atomic structure and the departure from Hooke's law in an approximate fashion. The result may or may not be sensitive to the details of the approximation. The force on an interstitial atom or dislocation is of the insensitive type; its value when Hooke's law is assumed to be obeyed even for infinite strains does not differ significantly from the value with a reasonable approximation to the actual non-linear behaviour (Koehler 1941; Leibfried 1949; Bilby 1950). This suggests that one ought to be able to introduce the concept of the force on a singularity quite generally into the classical theory of elasticity, so completing the parallel between elastic continuum and crystal lattice. The object of this paper is to show how this can be done and to devise a simple way of calculating the force on a given elastic singularity.

We have first to allow that singularities can move through the medium, a possibility not envisaged in the classical theory but not inconsistent with it. To see the lines on which the theory must then be developed it is useful to make a comparison with electrostatics. The force on a point charge is usually taken as the starting point, and development of the theory leads to the concept of an energy density. The total energy of a point charge is found to be infinite, but this does not cause difficulty until the more sophisticated electrodynamic problems are reached. In the elastic case things are reversed—we know the energy density and must infer the force. The problem of the infinite self-energy of point and line singularities must be faced from the outset. Alternatively, Poisson's equation can be taken as the starting point for electrostatics; point charges are then to be regarded as limiting cases in which the distribution of charge has the form of a delta-function. Similarly, we can develop the elastic theory for states of internal stress with finite total strain energy and regard point and line singularities as limiting cases. From this point of view the elastic analogue of the electric field produced by a continuous charge distribution is the general state of self-strain in which the stresses satisfy the equilibrium equations but (unlike stresses arising from body and surface forces) not the compatibility conditions. The elastic displacements cannot be defined everywhere, but their place can be taken by the three stress functions of Maxwell or Morera. These provide the analogue of the electrostatic potential; the counterpart of the charge is the 'incompatibility tensor' formed of the six expressions which when equated to zero yield the compatibility conditions. We shall, however, work with the conventional representation in terms of stress, strain and displacement even though this leads to a certain amount of manœuvring to circumvent the lack of a displacement function in regions where the incompatibility tensor is not zero.

In what follows, the term 'singularity' is usually to be taken to mean an extended state of internal stress of the kind just discussed, rather than a singularity in the mathematical sense.

To make this idea more concrete we first describe (§3) a general type of surface singularity with finite though discontinuous stresses and show how various point and line singularities can be derived as limiting forms of it.

To be able to speak of the movement of a singularity one must decide precisely what is meant by saying that two states of internal stress in the same body represent the same singularity in different positions. This is discussed in §4 and the result used (§5) to define provisionally and calculate the force which surface tractions exert on a singularity. The image force and the force which one singularity exerts on another are dealt with in §7 by a simple argument which also justifies the provisional definition of §5. These results apply only to homogeneous (but anisotropic) bodies. In §8 the force on an inhomogeneity in a body free of internal stress is defined and calculated. Section 9 ties up certain loose threads in the previous argument and extends the results to bodies containing both internal stresses and inhomogeneities, and acted on by body forces as well as surface tractions. Section 10 discusses possible extensions, in particular to the dynamical behaviour of singularities.

Apart from recent work inspired by the needs of metal physics the present problem received some attention at the end of the last century, when elastic solids furnished models for the ether. Larmor (1897) discussed an elastic singularity formed by cutting a lens-shaped cavity in the material, giving a relative rotation to the faces, and cementing them together. He pointed out that a pair of these singularities would exert forces on each other. In a remarkable paper, Burton (1892) considered the equations of motion of 'strain figures', states of stress which are possible without applied forces in a medium with a non-linear stress-strain relation. Burton (as he himself realized) was inconsistent in assuming superposability of stresses in his non-Hookean solid; his results can, however, be interpreted in terms of internal stresses in a Hookean medium.

2. Notation and integral theorems

Rectangular Cartesian co-ordinates are denoted by x_i $(i = 1, 2, 3)$. The elastic displacement vector has components u_i. Suffixes following a comma represent differentiation, so that, for example, $u_{i,j} = \partial u_i/\partial x_j$, $u_{i,jk} = \partial^2 u_i/\partial x_j \partial x_k$. A repeated suffix is to be summed over the values 1, 2, 3. The strain tensor e_{ij} and the stress tensor p_{ij} are defined by

$$\left.\begin{aligned} e_{ij} &= e_{ji} = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \\ p_{ij} &= p_{ji} = c_{ijkl}\,e_{kl} = c_{ijkl}\,u_{k,l} \end{aligned}\right\}. \tag{1}$$

The elastic modulus tensor c_{ijkl} (which may be a function of position) is unaltered by interchanging i with j or k with l or the pair (ij) with the pair (kl). In an isotropic body

$$p_{ij} = \lambda\, e_{kk}\,\delta_{ij} + 2\mu\, e_{ij}. \tag{2}$$

In equilibrium with no body forces the divergence of the stress-tensor vanishes:

$$p_{ij,j} = 0. \tag{3}$$

The traction on an element of area dS with normal n_j is $p_{ij}n_j\,dS = p_{ij}\,dS_j$, where dS_j is an abbreviation for $n_j\,dS$. For the e_{ij} to be derivable from a displacement according to (1) the six compatibility conditions $S_{rs} = 0$ must be satisfied, where

$$S_{rs}(e_{mn}) \equiv e_{rs,ii} + e_{ii,rs} - e_{ri,si} - e_{si,ri} - (e_{ii,jj} - e_{ij,ij})\,\delta_{rs}. \tag{4}$$

Let (u_i, e_{ij}, p_{ij}) and (u'_i, e'_{ij}, p'_{ij}) be two sets of quantities each related by (1) and satisfying (3). We have

$$p_{ij}\,e'_{ij} = p'_{ij}\,e_{ij} = p_{ij}\,u'_{i,j} = p'_{ij}\,u_{i,j} \tag{5}$$

by the symmetry properties of the c's. If we form the vector $\mathbf{v}$ with components

$$v_j = p_{ij}\,u'_i - p'_{ij}\,u_i,$$

it easily follows from (3) and (5) that $\operatorname{div}\mathbf{v} = v_{i,i} = 0$. Hence, by Gauss's theorem,

$$\int_{\Sigma_1} v_j\,dS_j = \int_{\Sigma_2} v_j\,dS_j \quad \text{or} \quad \int_{\Sigma_1 - \Sigma_2} v_j\,dS_j = 0, \tag{6}$$

where Σ_1 and Σ_2 are two closed surfaces which can be deformed into one another without passing through singularities of $\mathbf{v}$. If Σ is a closed surface containing no singularities of $\mathbf{v}$,

$$\int_{\Sigma} v_j\,dS_j = 0. \tag{7}$$

Equation (6) is essentially Betti's reciprocal theorem (Love 1927). We have assumed that $\mathbf{v}$ is single-valued. If it has to be made single-valued by a cut the two integrals in (6) differ by $\int v_j\,dS_j$ taken over both sides of the part of the cut surface intercepted between Σ_1 and Σ_2 (Colonnetti 1915).

If u'_i, p'_{ij} are corresponding displacement and stress, so are $u'_{i,l}, p'_{ij,l}$ as long as the body is homogeneous, and so by (6)

$$\int_{\Sigma_1 - \Sigma_2} (p_{ij}\,u'_{i,l} - p'_{ij,l}\,u_i)\,dS_j = 0 \quad \text{if} \quad c_{ijkl,m} = 0. \tag{8}$$

Again, if we have a continuous series of elastic states specified by a parameter ξ then $\partial u'_i/\partial\xi$, $\partial p'_{ij}/\partial\xi$ are corresponding displacement and stress if u'_i, p'_{ij} are, provided the c's are independent of ξ; then (6) gives

$$\int_{\Sigma_1 - \Sigma_2} (p_{ij}\,\partial u'_i/\partial\xi - u_i\,\partial p'_{ij}/\partial\xi)\,dS_j = 0 \quad \text{if} \quad \partial c_{ijkl}/\partial\xi = 0. \tag{9}$$

The following result will also be useful:

$$\int_{\Sigma} (w_{j,l} - w_{i,i}\,\delta_{jl})\,dS_j = -\int_{\sigma} \epsilon_{lij}\,w_i\,dx_j, \tag{10}$$

where Σ is a surface bounded by the curve σ, and w_i is single-valued and continuous. ϵ_{lij} is the completely antisymmetric tensor which is zero if two suffixes are equal and otherwise is $+1$ or -1 according as lij is an even or odd permutation of 123. Equation (10) is easily proved by applying Stokes's theorem to the tensor $\epsilon_{lij}\,w_i$ or by integrating $w_{i,i}$ over the volume generated by Σ during a small displacement parallel to the x_l-axis and then using Gauss's theorem. If w_i is multiple-valued on Σ but can be made single-valued by a cut C, we shall in general expect the integrand to have singularities at the ends of the cut. We can define the surface integral over Σ to be the limit of the integral over the region of Σ outside σ (figure 1 a) as the latter shrinks on to C. Then (assuming Σ is closed apart from the cut)

$$\int_{\Sigma} (w_{j,l} - w_{i,i}\,\delta_{lj})\,dS_j = -\int_{C} \epsilon_{lij}\,\Delta w_i\,dx_j, \tag{11}$$

provided the circles A, B make no contribution in the limit. (We shall usually be concerned with the case where C is a closed curve, or two-dimensional problems where C becomes a point on a plane curve.) Δw_i is the jump in w_i on crossing C, the sign being chosen so that if

$$\Delta w_i = \lim_{a \to b} \{w_i(a) - w_i(b)\}$$

the vectors $\overline{ab}$, dx_i and the outward-drawn normal form a right-handed system.

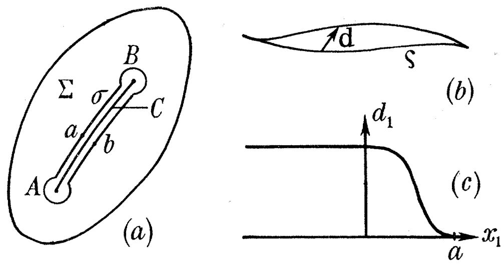

FIGURE 1

If we form a tensor $$t_{jl} = w_{j,l} - w_{i,i}\delta_{jl} \tag{12}$$
from any vector w_i, then for any closed surface Σ on which t_{jl} is single-valued

$$\int_{\Sigma} t_{jl}\, dS_j = 0,$$

even if there are singularities of t_{jl} inside Σ. This is to be contrasted with (7); in fact, $t_{jl,j}$ vanishes identically whilst $v_{j,j}$ does so only in virtue of the equation $p_{ij,j} = 0$.

3. Types of singularity

A large range of singularities can be regarded as particular cases of a type of surface singularity considered by Somigliana (1914) (cf. also Gebbia 1902; Mann 1949; Bogdanoff 1950). Following Neményi (1931) we shall call this singularity a Somigliana dislocation. It can be generated in the following way. Make a cut over a surface ς (open or closed) in an unstrained body and give the faces of the cut a small relative displacement, removing material where there would be interpenetration (figure 1 *b*). Fill in the remaining gaps and weld together. We are left with a system of internal strain which is completely characterized by giving as a function of position on ς the vector **d** which specifies the final separation of points originally adjacent on opposite sides of the cut. The stress $p_{ij}n_j$ (n_j is the normal to ς) is continuous across ς, but the individual components of p_{ij}, e_{ij} are in general discontinuous; these discontinuities can be calculated when **d** is known over ς. If **d** is a reasonably smooth function p_{ij} and e_{ij} will be finite everywhere except possibly near the edge of an open surface. (The condition for the displacement to be finite near the edge has been derived for the isotropic case by Gebbia.) Somigliana dislocations for which $\mathbf{d} = \mathbf{b} + \mathbf{r} \times \mathbf{c}$ with constant vectors **b** and **c** are the Weingarten dislocations discussed by Volterra. Dislocations for which $\mathbf{c} = 0$ are of particular interest; we shall call them physical dislocations. A number of physical dislocation loops lying in a surface (for example, near one of Frank & Read's (1950) dislocation sources) can be regarded as a general Somigliana dislocation for which **d** is a stepped function of position; at distances large compared with the spacing of the individual dislocations the stress is equivalent to that of a Somigliana dislocation with a continuously distributed **d**.

Near the boundary of a physical dislocation the stresses tend to infinity unless we remove the material in this region, as Volterra did. Alternatively, we can replace the physical dislocation by a Somigliana dislocation with **d** equal to **b** over most of ς but suitably tapered off near the boundary. Consider, for example, an infinite edge dislocation line along the x_3 axis with $\mathbf{d} = (b, 0, 0)$. For the tapered Somigliana dislocation we may, for example, take (figure 1*c*)

$$\begin{aligned} d_1 &= b(a^m - x_1^m)^n / a^{mn} \quad &(x_1 > 0) \\ &= b &(x_1 < 0) \end{aligned}$$

with the half-plane $x_2 = 0, x_1 < a$ for ς. From a known relation between stress and the discontinuity in displacement (Nabarro 1947; Eshelby 1949*b*) the shear stress p_{12} in the plane $x_2 = 0$ can be shown to be proportional to $x^{m-1}(a^m - x^m)^{n-1} a^{-mn} \log|1 - a/x_1|$, plus a polynomial in x_1. Hence by choosing m and n large enough we can make p_{12} and as many of its derivatives as we like continuous on the plane $x_2 = 0$. Further calculation shows that the same is true for all components of p_{ij} both on and off this plane. It is clear that a physical dislocation bounded by an arbitrary curve can be similarly regarded as the limit as $a \to 0$ of a suitable Somigliana dislocation dependent on a parameter a specifying the taper near the edge.

To derive point and line singularities from the general Somigliana dislocation we take a sphere or tube of radius r for ς and a suitable distribution of **d**. We then let r decrease to zero, at the same time increasing **d** in such a way that the displacement at a fixed point of observation remains finite. For example, if ς is a sphere and **d** is normal to ς and of magnitude $d = \text{const.}\, r^{-2}$ we obtain a centre of dilatation.

Thus physical dislocations and point and line singularities with infinite self-energy can all be regarded as limiting cases of Somigliana dislocations of finite self-energy.

As already explained, we may also have a volume distribution of internal stress in which (3) is satisfied, but the e_{ij} derived from the p_{ij} with the aid of Hooke's law do not satisfy $S_{ij} = 0$ everywhere. The regions in which $S_{ij} \neq 0$ are to be regarded as the actual seat of the singularity. This is discussed more fully in §9.

4. The stress-field of a singularity in a finite body

We need first to decide what is meant by a particular type of singularity in a body of given shape and size. A point singularity will naturally be defined as a solution of the elastic equations with vanishing $p_{ij} n_j$ at the surface of the solid and becoming infinite in a prescribed way at a certain point. In the general case it will be more convenient to regard each type of singularity as defined by giving its stress field in an infinite body and then prescribing a process for finding the stress field of the same singularity in a given finite body.

Draw a closed surface Σ_0 in an infinite homogeneous elastic medium. We shall say that there is a singularity inside Σ_0 if the stresses in it could not be produced by body forces outside Σ_0. We shall suppose that, when the region within a certain closed surface Σ_S has been excluded, p_{ij} and $u_{i,j}$ are continuous and single-valued and u_i is continuous but not necessarily single-valued in the rest of the interior of Σ_0.

For a point or line-singularity Σ_S can be taken as a small sphere or narrow tube surrounding the point or line on which the stress and displacement become infinite. If u_i or p_{ij} become infinite on or discontinuous across a surface we have a surface singularity, and for Σ_S we may take a jacket closely enveloping the singular surface. The conditions imposed on u_i and $u_{i,j}$ have been chosen so that physical dislocations can be regarded as line singularities rather than surface singularities. (If the physical dislocation is replaced by a tapered Somigliana dislocation, as described in § 3, Σ_S must enclose the whole of the tapered part.) For a volume singularity Σ_S must enclose the region in which $S_{rs} \neq 0$. We shall assume that the surface tractions over any surface within Σ_O and surrounding Σ_S are in rigid-body equilibrium. A distribution of body force inside Σ_S with zero resultant and moment therefore qualifies as a singularity according to our definition, though it is not usually what we have in mind.

Let the displacement and stress of a certain singularity in the infinite medium be $u_i^\infty(\mathbf{x}), p_{ij}^\infty(\mathbf{x})$. Then we define $u_i^\infty(\mathbf{x}-\boldsymbol{\xi}), p_{ij}^\infty(\mathbf{x}-\boldsymbol{\xi})$ to be the corresponding quantities for the singularity after it has been moved distances ξ_1, ξ_2, ξ_3 parallel to the x_1-, x_2- and x_3-axes. For a point singularity it is natural to take for the ξ_i the co-ordinates of the mathematical singular point. In general, though nothing more is implied than that variation of $\boldsymbol{\xi}$ translates the stress-system rigidly, it is still convenient to speak of the point $\boldsymbol{\xi}$ as the position of the singularity.

Starting from the case of an infinite medium we define the same singularity in a finite body in the following way. Remove the material outside Σ_O without allowing the surface forces $p_{ij}^\infty n_j$ on Σ_O to relax, so that the displacements and strains remain unaltered within Σ_O. Next reduce these surface forces to zero. We are left with a singularity in a body whose surface Σ_O is stress-free. The displacement and stress are

$$\left.\begin{aligned} u_i^S &= u_i^\infty(\mathbf{x}-\boldsymbol{\xi}) + u_i^I(\mathbf{x},\boldsymbol{\xi}), \\ p_{ij}^S &= p_{ij}^\infty(\mathbf{x}-\boldsymbol{\xi}) + p_{ij}^I(\mathbf{x},\boldsymbol{\xi}), \end{aligned}\right\} \tag{13}$$

where u_i^I and p_{ij}^I (which we may call the image displacement and stress) have no singularities within Σ_O and are such that $p_{ij}^S n_j = 0$ on Σ_O, i.e.

$$p_{ij}^I n_j = -p_{ij}^\infty n_j \quad \text{on} \quad \Sigma_O. \tag{14}$$

We have

$$u_{i,l}^\infty = -\partial u_i^\infty/\partial \xi_l, \quad p_{ij,l}^\infty = -\partial p_{ij}^\infty/\partial \xi_l. \tag{15}$$

There are no corresponding relations for u_i^I, p_{ij}^I, since they depend on $\mathbf{x}, \boldsymbol{\xi}$ not merely through the differences $(\mathbf{x}-\boldsymbol{\xi})$. However, from (14) and (13)

$$(\partial p_{ij}^I/\partial \xi_l)\, n_j = -(\partial p_{ij}^\infty/\partial \xi_l)\, n_j = p_{ij,l}^\infty n_j \text{ on } \Sigma_O. \tag{16}$$

5. The force exerted on a singularity by surface tractions

Now apply surface tractions $p_{ij}^A n_j$ to Σ_O, producing displacement and stress u_i^A, p_{ij}^A in the body in addition to the u_i^S, p_{ij}^S already present. Let the singularity undergo a translation $\delta\xi$ parallel to the x_l-axis. The work done by the surface tractions is

$$\delta W = \delta\xi \int_{\Sigma_O} p_{ij}^A \frac{\partial u_i^S}{\partial \xi_l}\, dS_j + O(\delta\xi^2).$$

We tentatively define

$$F_l^A = \lim_{\delta\xi\to 0} \frac{\delta W}{\delta\xi} = \int_{\Sigma_O} p_{ij}^A \frac{\partial u_i^S}{\partial \xi_l}\, dS_j \tag{17}$$

as the force which the surface tractions exert on the singularity. Because $p^S_{ij} n_j = 0$ on Σ_0 we can write

$$F^A_l = \int_\Sigma \left(p^A_{ij} \frac{\partial u^S_i}{\partial \xi_l} - u^A_i \frac{\partial p^S_{ij}}{\partial \xi_l} \right) dS_j$$

taken over the surface $\Sigma = \Sigma_0$. By (9) this integral can equally well be taken over any surface Σ in the body into which Σ_0 can be deformed without entering Σ_S. By (13)

$$F^A_l = \int_\Sigma \left(p^A_{ij} \frac{\partial u^\infty_i}{\partial \xi_l} - u^A_i \frac{\partial p^\infty_{ij}}{\partial \xi_l} \right) dS_j + \int_\Sigma \left(p^A_{ij} \frac{\partial u^I_i}{\partial \xi_l} - u^A_i \frac{\partial p^I_{ij}}{\partial \xi_l} \right) dS_j.$$

The second term vanishes by (9) and (7). Using (15) we have simply

$$F^A_l = \int_\Sigma (u^A_i p^\infty_{ij,l} - p^A_{ij} u^\infty_{i,l})\, dS_j, \tag{17'}$$

a form in which we do not need to know u^I_i, p^I_{ij}. This can be further transformed by (11). Since the only multiple-valued quantity which we allow is the displacement of a physical dislocation for which $\Delta u^S_i = \Delta u^\infty_i = \text{const.}$ we have

$$F^A_l = \int_\Sigma (p^A_{ij,l} u^\infty_i - u^A_{i,l} p^\infty_{ij})\, dS_j + \Delta u^\infty_k \int_C \epsilon_{lij} p^A_{ik}\, dx_j. \tag{18}$$

The further form

$$F^A_l = \int_\Sigma \{(\tfrac{1}{2} p^\infty_{ik} u^A_{i,k} \delta_{jl} - p^\infty_{ij} u^A_{i,l}) + (\tfrac{1}{2} p^A_{ik} u^\infty_{i,k} \delta_{jl} - p^A_{ij} u^\infty_{i,l})\}\, dS_j \tag{19}$$

is easily verified; it holds whether u^∞_i is single-valued or not.

6. The centre of dilatation: dislocations

The simplest singularity is a centre of dilatation in an isotropic body (Love 1927). If it is at the origin

$$u^\infty_i = \delta x_i / r^3 \quad \text{with} \quad r^2 = x_1^2 + x_2^2 + x_3^2, \tag{20}$$

where δ is a constant. To find the force on it we use (18), taking for Σ a small sphere of radius r about the origin. Expanding the applied stress and displacement in a Taylor series we have

$$F^A_l = p^A_{ij,l} \int u^\infty_i\, dS_j + p^A_{ij,lm} \int x_m u^\infty_i\, dS_j - u^A_{i,l} \int p^\infty_{ij}\, dS_j - u^A_{i,lm} \int x_m p^\infty_{ij}\, dS_j + \ldots,$$

where the A quantities are to be given their value at the origin. We have

$$\int u^\infty_i\, dS_j = \delta r^{-4} \int x_i x_j\, dS = \delta r^{-4} \delta_{ij} \int \tfrac{1}{2}(x_1^2 + x_2^2 + x_3^2)\, dS = \tfrac{4}{3}\pi \delta \delta_{ij}.$$

In a similar way we find, using (1) and (2),

$$\int x_m p^\infty_{ij}\, dS_j = -(16\pi/3)\, \mu \delta \delta_{mi}.$$

The term in $\int p^\infty_{ij}\, dS_j$ vanishes since the singularity is in equilibrium. The remaining terms are of order r. Since F^A_l is independent of the choice of r they can make no contribution whether we let r tend to zero or not. Thus

$$F^A_l = \tfrac{4}{3}\pi\delta(p^A_{ii,l} + 4\mu u^A_{i,il}) = 4\pi\delta\{(1-\sigma)/(1+\sigma)\} p^A_{ii,l},$$

where $\sigma = \lambda/2(\lambda+\mu)$ is Poisson's ratio. In general

$$\mathbf{F} = -12\pi\delta\{(1-\sigma)/(1+\sigma)\}\operatorname{grad} p^A, \tag{21}$$

where
$$p^A = -\tfrac{1}{3}(p^A_{11}+p^A_{22}+p^A_{33})$$

is the applied hydrostatic pressure at the position of the singularity.

The following model of an interstitial atom has been used by Bilby (1950). An elastic sphere of radius $(1+\alpha)r_0$ is forced into a spherical hole of radius r_0 in an infinite block of the same material. It can easily be shown (Mott & Nabarro 1940) that for $r > r_0$ the displacement is given by (20) with $\delta = \alpha r_0^3(1+\sigma)/3(1-\sigma)$, whilst for $r < r_0$ the material is uniformly compressed.

If we regard this as a volume singularity we may take for Σ any sphere of radius $r > r_0$. The force must then be given by (21), since this expression did not depend on the size of the surface over which we integrated. Alternatively, we may regard it as a Somigliana dislocation with a discontinuity αr_0 in the radial displacement across the sphere $r = r_0$. Σ must now be supplemented by a closed surface within the sphere $r = r_0$ in order that it may completely embrace the singular surface. Since this new surface can be contracted to a point it makes no contribution and the force is still given by (21). This is an exact result; F^A_l is proportional to the value of $\operatorname{grad} p^A$ at the centre of the sphere $r = r_0$, and there are no terms of order r_0 to be added.

For the two-dimensional problem of an infinite straight dislocation line along the x_3-axis with p^A_{ij}, u^A_i independent of x_3 we may use (18) to calculate F^A_l, the force per unit length, taking for Σ a cylinder of radius r and unit length with its axis along x_3. C is then unit length of a generator of this cylinder.

We could use explicit expressions for $u^\infty_i, p^\infty_{ij}$, but this would be tiresome in the anisotropic case. It is more interesting to see what is the least information we need about the dislocation in order to find F^A_l. The fundamental property is

$$\oint u_{i,j}\,dx_j = b_i \tag{22}$$

for any closed circuit embracing the dislocation line. This, however, does not completely define the singularity. Without altering (22) we could add single-valued line singularities (e.g. a line of dilatation) coincident with the dislocation line. They may be excluded by requiring that

$$\lim_{r\to 0} r u^\infty_i = 0, \tag{23}$$

where r is the distance of the point x_i from the singular line.

If we put $\Delta u^\infty_i = b_i$ and expand the A quantities about the origin (18) becomes

$$F^A_l = u^A_{i,l}\int_\Sigma p^\infty_{ij}\,dS_j + \epsilon_{lk3}p^A_{ik}b_i + O(r).$$

The first term vanishes since the dislocation is in equilibrium. Hence

$$F^A_1 = p^A_{2i}b_i, \quad F^A_2 = -p^A_{1i}b_i. \tag{24}$$

For a pure screw dislocation with $\mathbf{b} = (0, 0, b)$

$$F^A_1 = p^A_{23}b, \quad F^A_2 = -p^A_{13}b,$$

and for a pure edge dislocation with $\mathbf{b} = (b, 0, 0)$

$$F_1^A = p_{12}^A b, \quad F_2^A = -p_{11}^A b. \tag{24'}$$

In place of (24′) Koehler (1941) originally found in the isotropic case

$$F_1^A = \frac{2}{\pi}\frac{\lambda+\mu}{\lambda+2\mu} p_{12}^A b.$$

His method is equivalent to evaluating (17′) over a square surrounding the dislocation and omitting the first term in the integrand.

We can similarly show that for a loop of physical dislocation of arbitrary form

$$F_l^A = \lim_{r\to 0}\left\{\iint_\Sigma u_{i,l}^A p_{ij}^\infty \, dS_j + O(rl)\right\} + b_i \epsilon_{klj}\int_\sigma p_{ik}^A \, dx_j, \tag{25}$$

where Σ is a narrow tube of radius r and total length l embracing the singular line σ. The first term in (25) will vanish if we require that not only is $\int p_{ij}^\infty \, dS_j$ zero when taken over Σ but also when it is taken over any part of Σ intercepted between two planes perpendicular to the dislocation line (cf. Burgers 1939). Actually there is no harm in including in the surface of integration the two cross-sections of the tube, since according to (23) p_{ij}^∞ ultimately behaves like r^{-1}. An equivalent condition for the vanishing of the surface integral in (25) is thus that any element of volume is in static equilibrium whether it is traversed by the singular line or not. If this is admitted,

$$F_l^A = b_1 \epsilon_{klj}\int_\sigma p_{ik}^A \, dx_j.$$

This is consistent with a force $\epsilon_{klj} b_i p_{ik}^A s_j$ per unit length on a dislocation line at a point where its unit tangent vector has components s_j (Peach & Koehler 1950; Nabarro 1951). We cannot prove this by the present method, which only considers translations of the loop without change of form.

For the general Somigliana dislocation

$$F_l^A = \int_\varsigma d_i p_{ij,l}^A \, dS_j,$$

with the ς and $\mathbf{d}$ of § 3.

7. The image force and the forces between singularities

Consider a body containing a singularity S whose energy-density $\mathscr{W} = \frac{1}{2} p_{ij}^S u_{i,j}^S$ is everywhere finite. The internal elastic energy of the body,

$$W_{\text{int.}} = \int \mathscr{W} \, dv,$$

is a function of the parameters ξ_i defining the position of the singularity. We may regard

$$F_l^I = -\partial W_{\text{int.}}/\partial \xi_l,$$

which measures the rate of decrease of $W_{\text{int.}}$ when the singularity is displaced in the x_l direction, as the force acting on it in the absence of applied forces or other singularities. Since F_l^I depends on the existence of the free surface it will be called the image force.

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If there is a second singularity T whose energy-density is likewise finite everywhere we shall now have $\mathscr{W} = \frac{1}{2}(p^S_{ij} + p^T_{ij})(u^S_{i,j} + u^T_{i,j})$, and if S is moved whilst T remains fixed, $-\partial W_{\text{int.}}/\partial \xi_l$ will have a value differing from F^I_l. If

$$-\partial W_{\text{int.}}/\partial \xi_l = F^I_l + F^T_l, \tag{26}$$

we can regard F^T_l as the force which T exerts on S. Similarly, if there are in addition constant externally applied stresses we shall find, say,

$$-\partial W_{\text{int.}}/\partial \xi_l = F^I_l + F^T_l + F^{A\prime}_l.$$

If $W_{\text{ext.}}$ is the potential energy of the source of the applied stress, $-\partial W_{\text{ext.}}/\partial \xi_l$ is F^A_l as defined in §5. The total force on the singularity S can then be defined as

$$F_l = -\partial(W_{\text{int.}} + W_{\text{ext.}})/\partial \xi_l = F^I_l + F^T_l + F^A_l + F^{A\prime}_l,$$

and $F_l \delta\xi$ is the decrease in energy of the whole system when S is displaced from ξ_i to $\xi_i + \delta_{il}\delta\xi$. Consequently our previous definition of F^A_l is reasonable only if we can show that $F^{A\prime}_l$ vanishes. To calculate F^I_l and F^T_l in a simple way for a homogeneous medium we use this known result:

An elastic body reacts to applied forces in the same way whether it is self-stressed or not. (27)

(See, for example, Southwell (1936) and the discussion and references in Neményi (1931). Gebbia (1902) stated (27) for the case of Somigliana dislocations.)

The following discussion is valid for singularities with finite self-energy. To apply the results to singularities with infinite self-energy we must make the following explicit assumption:

Singularities with infinite self-energy can be regarded as limiting cases of singularities with finite self-energy, and when we make the passage to the limit the expression for the force is still valid.

Alternatively we can simply lay down these two axioms:

(i) The statement (27) is true also for singularities with infinite self-energy.

(ii) The plausible subtraction of infinities implicit in equation (28) below is allowable.

As in §4 we take the singularity in an infinite medium, describe a surface Σ_0 around it and cut out the body bounded by Σ_0 without allowing the surface forces to relax. In this way we get a body with the singularity in the position specified by the parameters ξ_i. If we had translated Σ_0 a distance $-\delta\xi$ parallel to the x_l-axis before cutting it, we should have got a body equivalent to the first but with the singularity in the position specified by the parameters $\xi_i + \delta_{il}\delta\xi$. The difference of energy in these two cases before the surface tractions are relaxed is (figure 2*a*)

$$\frac{1}{2}\int_{\Sigma_0} p^\infty_{ik} e^\infty_{ik}\, dS_l\, \delta\xi + O(\delta\xi^2). \tag{28}$$

(It is here that any necessary 'subtraction of infinities' occurs; energy in the shaded region of figure 2*a* makes no contribution.) When the surface tractions are relaxed in the first case the elastic energy is reduced by an amount

$$-\frac{1}{2}\int_{\Sigma_0} p^\infty_{ij}\, u^I_i\, dS_j.$$

(Here we invoke (27).) The corresponding quantity in the second case is

$$-\left(1 + \delta\xi \frac{\partial}{\partial \xi_l}\right)\frac{1}{2}\int_{\Sigma_0} p^\infty_{ij}\, u^I_i\, dS_j + O(\delta\xi^2).$$

Consequently, after relaxing the surface tractions the energy for the first case less that for the second is

$$\frac{1}{2}\int_{\Sigma_0}\{p^{\infty}_{ik}e^{\infty}_{ik}\delta_{lj}-\partial(p^{\infty}_{ij}u^I_i)/\partial\xi_l\}\,dS_j\,\delta\xi+O(\delta\xi^2).$$

This must be equal to $F^I_l\,\delta\xi+O(\delta\xi^2)$. Rearranging and using (11) and (15) we have

$$F^I_l=\int_{\Sigma_0}p^I_{ij}\frac{\partial u^S_i}{\partial\xi_l}dS_j+\frac{1}{2}\int_{\Sigma_0}\left(\frac{\partial p^I_{ij}}{\partial\xi_l}u^I_i-p^I_{ij}\frac{\partial u^I_i}{\partial\xi_l}\right)dS_j+\int_{\Sigma_0}(\tfrac{1}{2}p^{\infty}_{ik}u^{\infty}_{i,k}\delta_{jl}-p^{\infty}_{ij}u^{\infty}_{i,l})\,dS_j.$$

The second term vanishes by (7) and (9). According to (8) the last term is unchanged if Σ_0 is deformed in any way as long as it still encloses the singularity. Hence this term is a constant characteristic of the singularity in an infinite medium and entirely independent of the size and shape of the body containing it. When we let Σ_0 tend to infinity the term vanishes if u^{∞}_i behaves like r^{-1} or $\log r$ at large distances in a three-dimensional or two-dimensional problem. For isotropy the theory of biharmonic functions shows that these conditions are fulfilled if the stresses do in fact fall off at all with r, as they must do if there is not to be a singularity at infinity. It is fairly obvious that the same will be true for anisotropy. We return to this point in § 9 and meanwhile assume that the three quantities

$$F^{\infty}_l=\int_{\Sigma}(\tfrac{1}{2}p^{\infty}_{ik}u^{\infty}_{i,k}\delta_{jl}-p^{\infty}_{ij}u^{\infty}_{i,l})\,dS_j \tag{29}$$

vanish for any surface Σ surrounding the singularity. Manipulating the surviving integral in F^I_l as we did (17), we have, for example,

$$F^I_l=\int_{\Sigma}(u^I_i\,p^{\infty}_{ij,l}-p^I_{ij}\,u^{\infty}_{i,l})\,dS_j.$$

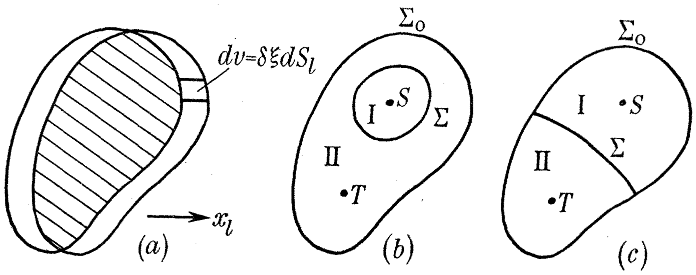

FIGURE 2

The case where there is a fixed singularity T can be treated by a slightly more elaborate set of imaginary operations. (We shall not trouble to insert terms of order $\delta\xi^2$.) Draw a surface Σ in Σ_0 surrounding S but not T, thus dividing the body into parts I and II (figure 2*b*). (The final result is the same with the scheme of figure 2*c*, where it is easier to imagine separating I from II, but the argument becomes more prolix if Σ and Σ_0 are not separate.) Let u_i be the total displacement and put $u'_i=u_i-u^{\infty}_i=u^I_i+u^T_i$ and similarly for p_{ij} and p'_{ij}. Carry out the following steps:

(i) Move S from ξ_i to $\xi_i+\delta_{il}\,\delta\xi$ and record the new surface tractions

$$\{p_{ij}+\delta\xi(\partial p_{ij}/\partial\xi_l)\}\,n_j \tag{30}$$

across Σ. Move S back to ξ_i.

(ii) Cut the surface Σ and remove I without allowing the tractions on the surface of I and the surface of the cavity in II to relax. Alter the surface tractions on I from $p_{ij}n_j$ to $p^\infty_{ij}n_j$. The energy entering I in this process is

$$\frac{1}{2}\int_\Sigma (p_{ij}+p^\infty_{ij})\,(u^\infty_i-u_i)\,dS_j = -\int_\Sigma (p^\infty_{ij}+\tfrac{1}{2}p'_{ij})\,u'_i\,dS_j.$$

(iii) Change the position of S from ξ_i to $\xi_i+\delta_{il}\delta\xi$ and adjust the surface tractions in I to $\{p^\infty_{ij}+\delta\xi(\partial p^\infty_{ij}/\partial\xi_l)\}n_j$. The increase of energy in I is obviously given by (28) with Σ in place of Σ_0.

(iv) Alter the surface traction on I to (30). The energy entering I is

$$(1+\delta\xi\,\partial/\partial\xi_l)\int_\Sigma (p^\infty_{ij}+\tfrac{1}{2}p'_{ij})\,u'_i\,dS_j.$$

(v) Alter the surface traction on the cavity in II to (30). The energy entering II is

$$-\delta\xi\int_\Sigma p_{ij}(\partial u_i/\partial\xi_l)\,dS_j.$$

The minus sign is correct if the normal is directed from the cavity into II.

(vi) The stress and displacement are now equal at corresponding points of the surface of I and the cavity in II. Replace I in the cavity and rejoin the surface Σ. There is no energy change in I or II.

Adding the different energy changes and using (29) we have

$$-\frac{\partial W}{\partial\xi_l} = \int_\Sigma\left\{(p^T_{ij}+p^I_{ij})\frac{\partial u^\infty_i}{\partial\xi_l}-\frac{\partial p^\infty_{ij}}{\partial\xi_l}(u^T_i+u^I_i)\right\}dS_j+\frac{1}{2}\int_\Sigma\left(p'_{ij}\frac{\partial u'_i}{\partial\xi_l}-\frac{\partial p'_{ij}}{\partial\xi_l}u'_i\right)dS_j, \tag{31}$$

and this must equal $F^T_l+F^I_l$. The second integral vanishes by the usual argument, and from (15) and (26) we have

$$F^T_l = \int_\Sigma (p^\infty_{ij,l}u^T_i-p^T_{ij}u^\infty_{i,l})\,dS_j.$$

Finally, let us repeat the previous argument for the case where there are also applied surface tractions $p^A_{ij}n_j$ on Σ_0. The only differences are first that the primed quantities must be interpreted as

$$u'_i = u^I_i+u^T_i+u^A_i, \quad p'_{ij} = p^I_{ij}+p^T_{ij}+p^A_{ij},$$

and secondly that in stage (v) the displacement on Σ_0 alters from u_i to $u_i+\delta\xi(\partial u_i/\partial\xi_l)$ and the surface forces do work. The following terms must thus be added to (31):

$$\int_\Sigma\left(p^A_{ij}\frac{\partial u^\infty_i}{\partial\xi_l}-\frac{\partial p^\infty_{ij}}{\partial\xi_l}u^A_i\right)dS_j-\int_{\Sigma_0}p^A_{ij}\frac{\partial u_i}{\partial\xi_l}\,dS_j.$$

But the first term is simply F^A_l as defined in § 5, and the second is $-F^A_l$ since

$$\partial u_i/\partial\xi_l = \partial(u^\infty_i+u^I_i)/\partial\xi_l = \partial u^S_i/\partial\xi_l,$$

u^T_i and u^A_i remaining constant. Consequently the presence of an externally-applied stress makes no difference at all to the change of internal energy when the singularity is displaced. Thus $F^{A\prime}_l$ vanishes as we required above.

Let $\phi_l(A,\infty)$ denote one of the expressions for F^A_l in § 5 or the explicit expression for a particular singularity. Then clearly $F^I_l = \phi_l(I,\infty)$, with an obvious notation. Thus if surface tractions equal and opposite to $p^I_{ij} n_j$ are applied there is no total force on the singularity: $F^A_l + F^I_l = 0$. This can be described picturesquely by saying that since the total stress and displacement are p^∞_{ij} and u^∞_i the singularity thinks that it is in neutral equilibrium in an infinite medium with no applied stress. Similarly, $F^T_l = \phi_l(T, \infty)$, with the proviso that when $\phi_l(T, \infty)$ takes the form of a surface integral the surface of integration Σ shall separate S from T.

The expression for the total force is

$$F_l = F^A_l + F^I_l + F^T_l = \int_\Sigma P_{jl}\, dS_j, \tag{32}$$

where

$$P_{jl} = -p_{ij} u_{i,l} + \tfrac{1}{2} p_{ik} e_{ik} \delta_{jl}.$$

This easily follows by taking ϕ_l in the form (19), adding F^∞_l and noting that by (3) and (11)

$$\int_\Sigma (p'_{ij} u'_{i,l} - \tfrac{1}{2} p'_{ik} e'_{ik} \delta_{jl})\, dS_j = 0,$$

with $u'_i = u^A_i + u^I_i + u^T_i$, etc. Since we have reintroduced F^∞_l the truth of (32) does not depend on its vanishing. S and T can of course each be a group of singularities. Consequently (32) gives the force on all the singularities within Σ exerted by

(i) their image stresses,

(ii) externally applied stresses,

(iii) all the singularities outside Σ.

By analogy with electrostatics we may call P_{jl} the Maxwell tensor of elasticity. Note that P_{jl} is not symmetric; l specifies the component of the force and j is used to form an inner product with the surface element. In an infinite medium free from applied stresses the force which one group of singularities exerts on a second group is equal and opposite to the force which the second exerts on the first. More generally action and reaction are equal and opposite in the following sense: the total force on the singularities in I (figure 2*c*) plus the total force on the singularities in II is equal to the total force on all the singularities within Σ_0, the integrals over the partition Σ cancelling.

The force which one infinite straight dislocation exerts on another parallel to it can be found from (24) with b_i equal to the Burgers vector of one dislocation and p^A_{ij} equal to the stress produced by the other. For a pair of edge dislocations the results agree with those of Leibfried (1949) and Bilby (1950) but not with the original calculations of Koehler (1941). If Koehler's calculation is repeated in bipolar co-ordinates the result agrees with Leibfried, Bilby and the present paper. The error appears to arise from the fact that Koehler had to split the total energy into self-energy and interaction terms which are difficult to evaluate over the same area. It is easy, for example, to evaluate the former over a circle and the latter over a rectangle. When we let these two areas of integration tend to infinity the result obtained depends on their relative size and shape. Instead of neglecting the energy within small cylinders surrounding the dislocation lines we may cut out these cylinders leaving stress-free holes at the centres of the dislocations, in the manner of Volterra. This problem can be solved rigorously by an extension of the analysis of Dean & Wilson (1947). The result differs from that obtained from (24) by terms which vanish with vanishing radius of the stress-free cylinders. The additional terms represent the image force on one dislocation due to the hole at the other.

It should be noted that singularities with single-valued displacements cannot have their centres excluded by stress-free surfaces in this way. For example, we can only annul the traction over a sphere surrounding a centre of dilatation by superimposing a hydrostatic pressure (giving the wrong behaviour at infinity) or an equal centre of compression at the same point, in which case the stress is zero everywhere.

The force exerted on a centre of dilatation by a dislocation, obtained by replacing p^A in (21) by the hydrostatic pressure of the dislocation, agrees with Bilby's result. It may be noted that Cottrell's (1949) process of creating one singularity in the stress field of another also gives the correct result. To create Bilby's model of an interstitial atom we have to make a spherical cut of radius r_0 and blow it up into a spherical annulus of thickness αr_0 (in the notation of §6) against the applied stress. This leads at once to an interaction energy of which (21) is the gradient. We defer discussing Leibfried's result for this case to §9 since it involves elastic inhomogeneities.

Even when F_l^A is known for a certain singularity the determination of F_l^I still requires the solution of a boundary-value problem to find p_{ij}^I. In some two-dimensional isotropic problems we can use existing solutions of related problems.

For a screw dislocation at the origin in a cylinder whose cross-section is bounded by the curve Σ_0 we must find a harmonic function u_3 for which $\partial u_3/\partial n$ vanishes on Σ_0 (no surface traction) and behaving like $\tan^{-1}(x_2/x_1)$ near the origin. The logarithmic potential, ϕ, of a charge at the origin inside a curve Σ_0 at constant potential is harmonic, satisfies $\partial\phi/\partial s = 0$ on Σ_0 and behaves like $\log r$ near the origin. Clearly u_3 is the harmonic function conjugate to ϕ. The conjugate function need not actually be constructed if only the stresses are required, since $p_{23}/\mu = u_{3,2} = \phi_{,1}$ and $p_{31}/\mu = u_{3,1} = -\phi_{,2}$. The case of a screw dislocation in a rectangular prism can be dealt with by the analysis of Courant & Hilbert (1931, p. 333; see also Leibfried & Dietze 1949). As a simpler case we see that the image stress for a screw dislocation distant r from the centre of a cylinder of radius R is given by an image dislocation of opposite sign distant R^2/r from the centre and on the same radius. The image force is $(\mu b^2/2\pi)\, r/(R^2-r^2)$, directed radially outwards. More generally the force on a screw dislocation is the same (in suitable units) as that on the line-charge in the associated electrostatic problem.

Again, the deflexion w of a plate and the Airy stress function χ of a plane-strain problem both satisfy the biharmonic equation with the boundary conditions $w = 0$, $\partial w/\partial n = 0$ and $\partial(\partial\chi/\partial x_1)/\partial s = 0$, $\partial(\partial\chi/\partial x_2)/\partial s = 0$ at a clamped edge and free surface respectively. The boundary conditions can be made to coincide if the plate is given a small rotation, or a linear expression in x_1, x_2 is added to χ, which does not alter the stresses. Near a concentrated load at $\boldsymbol{\xi}$, w behaves like $(\mathbf{x}-\boldsymbol{\xi})^2\log(\mathbf{x}-\boldsymbol{\xi})^2$. For χ this singularity represents a Weingarten dislocation made by inserting a narrow wedge with its apex at $\boldsymbol{\xi}$. For an edge dislocation at the same point with Burgers's vector $\mathbf{b}$, χ behaves like $\mathbf{b}'.(\mathbf{x}-\boldsymbol{\xi})\log(\mathbf{x}-\boldsymbol{\xi})^2$ with $\mathbf{b}'$ perpendicular to $\mathbf{b}$. Hence if w is the deflexion for a plate clamped around the curve Σ_0 and loaded at (ξ_1, ξ_2), then $\chi_W = \text{const.}\, w$ and

$$\chi_E = \text{const.}\left(b_1\frac{\partial}{\partial\xi_2} - b_2\frac{\partial}{\partial\xi_1}\right)w$$

are the respective stress functions for a Weingarten dislocation or an edge dislocation with $\mathbf{b} = (b_1, b_2)$ at (ξ_1, ξ_2) in a cylinder with Σ_0 as stress-free boundary. The results of Michell for a disk (Love 1927, p. 491) can be used to discuss an edge dislocation in a circular cylinder. Koehler (1941) has treated the case where the slip plane is a diameter of the cylinder.

As a three-dimensional example consider the image force urging a centre of dilatation towards the surface of a semi-infinite solid. The shear stress over the plane $x_3 = 0$ due to a centre of dilatation at $(0, 0, Z)$ in an infinite medium can be removed by putting an equal centre of dilatation at $(0, 0, -Z)$. This does not contribute to the image force since it produces a pure shear. The effect of annulling the remaining normal pressure on the plane $x_3 = 0$ can be found by integration from the expression (Love 1927, p. 191) for the effect of a concentrated load on the surface of a semi-infinite body. We find

$$\left(\frac{\partial p^I_{ii}}{\partial x_3}\right)_{x_3=Z} = \frac{3\mu(1+\sigma)\,\delta}{2Z^4},$$

so that by (21)

$$F^I = 6\pi\mu(1-\sigma)\,\delta^2 Z^{-4}$$

directed towards the free surface. For the Bilby singularity of § 6

$$F^I = \tfrac{1}{3}\pi\alpha^2 E r_0^2 \frac{1+\sigma}{1-\sigma}\left(\frac{r_0}{Z}\right)^4,$$

where E is Young's modulus and Z is the distance of the centre of the sphere below the free surface. This force can be regarded as derived from a potential

$$U = -\tfrac{1}{9}\pi\alpha^2(E r_0^3)\,\frac{1+\sigma}{1-\sigma}\left(\frac{r_0}{Z}\right)^3.$$

If we apply these results to an interstitial atom or lattice vacancy we see that in thermal equilibrium the concentration of these defects should be greater near the surface than in the body of the material. However, even with a fairly generous choice of the misfit constant α, U is of the order of kT at room temperature only when Z is of the order of r_0. At this point the approximation of an elastic continuum breaks down. When we reach a depth at which the continuum approximation is reasonable the concentration will have reached substantially its bulk value.

Two centres of dilatation in an infinite medium do not interact with one another (so long as linear isotropic elastic theory is valid), since each produces a pure shear stress. On the other hand, in the neighbourhood of a free surface each of them is acted on by its own and the other's image stress. It is clear that each one of a group of n centres of dilatation close together in comparison with their distance from the surface will be acted on by n times the force it would experience in isolation. However, if one centre is kept fixed at $(0, 0, Z)$ and the image force on it due to a second centre at (x_1, x_2, x_3) is averaged for all x_1, x_2 the result is zero, and so there is no long-range multiplicative effect.

8. The force on an inhomogeneity

Suppose that the elastic constants of an inhomogeneous body free from internal stress depend on three parameters ξ_i according to the relation

$$c_{ijkm} = c_{ijkm}(x_n - \xi_n). \tag{33}$$

The ξ_i might, for example, be the co-ordinates of a foreign inclusion in an otherwise uniform body. If fixed surface tractions are applied to the surface (so that $\partial(p_{ij}n_j)/\partial\xi_l = 0$ on Σ_0) and one of the parameters is changed, we have with the notation of §7

$$-\frac{\partial W_{\text{ext.}}}{\partial \xi_l} = \int_{\Sigma_0} p_{ij}\frac{\partial u_i}{\partial \xi_l}dS_j = \frac{\partial}{\partial \xi_l}\int p_{ij}e_{ij}\,dv = 2\frac{\partial W_{\text{int.}}}{\partial \xi_l},$$

so that of the work done by the external forces in a small change of ξ_l half disappears and half remains as an increase of the internal energy of the body. Consequently we can define

$$F_l = +\partial W_{\text{int.}}/\partial \xi_l$$

as the force which the applied surface tractions exert on the inhomogeneity.

We have

$$\frac{\partial W_{\text{int.}}}{\partial \xi_l} = -\frac{1}{2}\int \frac{\partial c_{ijkm}}{\partial \xi_l}e_{ij}e_{km}\,dv. \tag{34}$$

This can be seen as follows. The difference of energy between two bodies of the same size and shape (distinguished by unprimed and primed quantities) with different inhomogeneous elastic constants and acted on by the same surface tractions is

$$\delta W = \frac{1}{2}\int (p'_{ij}e'_{ij} - p_{ij}e_{ij})\,dv.$$

But

$$\int (p_{ij} - p'_{ij})\,e_{ij}\,dv = \int_{\Sigma_0}(p_{ij} - p'_{ij})\,u_i\,dS_j = 0,$$

since $p_{ij}n_j = p'_{ij}n_j$ at the surface. Hence we can replace $p_{ij}e_{ij}$ by $p'_{ij}e_{ij}$ in δW and similarly $p'_{ij}e'_{ij}$ by $p_{ij}e'_{ij}$. Hence

$$\delta W = \frac{1}{2}\int (c_{ijkm} - c'_{ijkm})\,e'_{ij}e_{km}\,dv,$$

from which (34) follows. From (33) and (11)

$$F_l = \frac{1}{2}\int c_{ijkm,l}u_{i,j}u_{k,m}\,dv$$

$$= \frac{1}{2}\int \{(c_{ijkm}u_{i,j}u_{k,m})_{,l} - 2c_{ijkm}u_{i,j}u_{k,lm}\}\,dv$$

$$= \int_{\Sigma_0}(\tfrac{1}{2}p_{km}u_{k,m}\delta_{lj} - p_{jk}u_{k,l})\,dS_j,$$

so that F_l is given by the 'Maxwell tensor'. If the body is homogeneous except in a limited region, Σ_0 can be replaced by a surface Σ enclosing that region.

The stress and displacement can be written as

$$p_{ij} = p^A_{ij} + p^S_{ij}, \quad u_i = u^A_i + u^S_i,$$

where u^A_i is the displacement which the given surface tractions would produce in the body if it were homogeneous throughout, and u^S_i can be regarded as the disturbance produced by the inhomogeneity. The stresses p^A_{ij} could only occur in the actual body if there were body forces of density $f_i = -(c_{ijkm}u^A_{k,m})_{,j}$ present. Consequently, the displacement and stress are the same as would be produced by body forces $-f_i$ in the inhomogeneous body. f_i vanishes where the body is homogeneous and the volume integral of f_i is zero. We may say that the surface forces 'induce' in the inhomogeneity a singularity specified by u_i^S, p_{ij}^S and then exert a force on it. We can write $u_i^S = u_i^\infty + u_i^I$ as for a real singularity, and similarly for p_{ij}^S; u_i^∞ is the displacement which the body forces $-f_i$ would produce if the body were continued to infinity, the additional material being homogeneous. F_l^∞ (equation (29)) clearly vanishes, and so we can write

$$F_l = \phi_l(A, \infty) + \phi_l(I, \infty),$$

just as for a real singularity.

As a simple example consider a body with shear and bulk moduli μ and K containing a spherical inclusion of radius r_0 and bulk modulus K'. It is easy to show that, under a uniform hydrostatic pressure p, u_i^S is given by (20) with

$$\delta = p\frac{r_0^3}{4\mu}\frac{3(K'-K)}{4\mu+3K'}$$

if the body is infinite. If p is not constant F_l will be given approximately by (21) with this value of δ. This is inaccurate because the variation of p will induce higher order singularities and the result must be multiplied by $1+O(r_0 \,|\, \mathrm{grad}\, p \,|/p)$. We have also neglected the image terms, which will be small if the inhomogeneity is far from the surface of the solid. For a spherical cavity ($K' = 0$) the approximate result is

$$\mathbf{F} = \tfrac{3}{2}\pi\frac{r_0^3}{\mu}\frac{\sigma}{1-\sigma}\,\mathrm{grad}\,(p^A)^2.$$

Such a cavity could migrate by evaporation and condensation or surface migration of molecules in the cavity or by diffusion of defects in the body of the solid. Similarly, for an incompressible inclusion ($K' = \infty$)

$$\mathbf{F} = -\tfrac{3}{2}\pi\frac{r_0^3}{\mu}\frac{1-\sigma}{1+\sigma}\,\mathrm{grad}\,(p^A)^2. \tag{35}$$

9. The general case

It would be tedious to treat the general case, where there are both singularities and inhomogeneities, by the methods of §7, 8. We shall therefore definitely adopt the point of view that singularities can be regarded as limiting cases of extended states of internal stress. We can then carry out volume integration without scruple; the only difficulty is the impossibility of defining a displacement function everywhere.

In order to make clear the nature of these internal stresses we consider a process (Timoshenko 1934) which would actually produce them. Reissner (1931) gives a more formal account.

Cut an unstrained body into infinitesimal cubes and give each one a permanent strain e_{ij}^*, for example by plastic deformation or by adding and removing material on its faces. Apply stresses $-c_{ijkl}e_{kl}^*$ to them; they become cubes again and can be welded together in the same relative position to give the original body. Since the surface stresses thus built in are not generally equal and opposite on the adjacent faces of neighbouring cubes, we are left with a distribution of body force of amount $(c_{ijkl}e_{kl}^*)_{,j}$ per unit volume. If we superimpose a distribution of body force

$$f_i = -(c_{ijkl}e_{kl}^*)_{,j}, \tag{36}$$

we are left with a body free of body forces but in a state of self-stress p^S_{ij} given by

$$p^S_{ij} = -p^*_{ij} + p^\dagger_{ij},$$

where $p^*_{ij} = c_{ijkl}\, e^*_{kl}$ and $p^\dagger_{ij}$ is a solution of $p^*_{ij,j} + (c_{ijkl}\, e^*_{kl})_{,j} = 0$ chosen so that p^S_{ij} satisfies the prescribed boundary conditions. Such a state of self-stress satisfies the equilibrium equations $p^S_{ij,j} = 0$, but the strains derived from it by Hooke's law do not satisfy the compatibility conditions since according to (4) $S_{ij}(e^S_{kl}) = S_{ij}(e^*_{kl}) \neq 0$, the e^*_{ij} being arbitrary functions. In regions where $e^*_{ij} = 0$ and hence $p^*_{ij} = 0$ the state of the medium is indistinguishable from that produced by body forces (36) and a displacement function can be defined. Where $S_{ij} \neq 0$ the stresses cannot be imitated by body forces. Suppose that S_{ij} is known as a function of position. Then if we use Hooke's law to replace the e's by p's in (4) we get the Michell-Beltrami compatibility equation modified by the presence of the term S_{ij}. This inhomogeneous equation together with the equilibrium equations and the boundary conditions determines the stress uniquely, just as in the homogeneous case.

To the first order in the e^*_{ij}, $R_{prst} = \epsilon_{ipr}\epsilon_{jst} S_{ij}$ is the Riemann tensor derived from the metric $g_{ij} = \delta_{ij} + 2e^*_{ij}$. Formally, this means that our set of deformed cubes could be fitted together without stress in the non-Euclidean space defined by the g_{ij}. This is quite intelligible in the two-dimensional case. Let a thin lamina be cut into small squares each of which is given a permanent strain $e^*_{11}, e^*_{22}, e^*_{12}$. The pieces could be fitted together without stress as a mosaic on a surface with Gaussian curvature $K = 2\, e^*_{12,12} - e^*_{11,22} - e^*_{22,11}$ at each point. Conversely, a self-stressed lamina can relieve its stress by buckling into a suitable surface.

When e^*_{ij} is known the displacement corresponding to $p^\dagger_{ij}$ in a homogeneous medium is clearly

$$u^\dagger_m(\mathbf{x}') = c_{ijkl} \int u_{mi}(\mathbf{x}' - \mathbf{x})\, e^*_{kl,j}\,(\mathbf{x})\, dv,$$

where $u_{mi}(\mathbf{x})$ is the displacement at $\mathbf{x}$ produced by a unit concentrated force in the x_i direction at the origin. u_{mi} can be found from the six equilibrium equations $c_{ijkl} u_{kn,lj} = \delta_{in}\delta(\mathbf{x})$ by taking their Fourier transforms and solving the resulting algebraic equations. The result is

$$u_{kn}(\mathbf{x}) = -(2\pi)^{-\frac{3}{2}} \iiint_{-\infty}^{\infty} K_{nk} \exp(-ik_i x_i)\, dk_1\, dk_2\, dk_3,$$

where K_{ik} is the matrix reciprocal to $c_{ijkl} k_j k_l$. By changing to the dimensionless integration variables $l_i = k_i r$, where r is the distance from 0 to $\mathbf{x}$, it is easy to show that u_{kn} is the product of r^{-1} and a function of direction. Since $u^\dagger_i$ is in fact the actual displacement u^∞_l in regions where $S_{ij} = 0$, this verifies the vanishing of F^∞_l assumed in § 7.

For the type of internal stress we are considering the assumption (27) follows immediately. The energy of a body subject to external and self-stresses p^A_{ij} and p^S_{ij} differs from the sum of the energies it would have if p^A_{ij} or p^S_{ij} were present separately by the interaction term $\frac{1}{2}\int (p^S_{ij} e^A_{ij} + p^A_{ij} e^S_{ij})\, dv$. The first term can be written as $\frac{1}{2}\int (p^A_{ij} u^A_i)_{,j}\, dv = \frac{1}{2}\int p^S_{ij} u^A_i\, dS_j = 0$, since $p^S_{ij} n_j = 0$ on the surface. The second interaction term cannot be treated in the same way, since e^S_{ij} cannot be derived from a displacement everywhere. However, by (5) it is equal to the first term. Hence the interaction term is zero. The behaviour of the body towards applied forces is determined by the way its energy depends on $p^A_{ij} n_j$ and (27) follows.

Similarly, we can express the interaction energy between two singularities as an integral over a surface separating them. In figure 2*b*, u_i^S exists in II and u_i^T in I but not vice versa. We can, however, express the interaction energy in I in terms of u_i^T and in terms of u_i^S in II. By applying Gauss's theorem we find the interaction energy in the form

$$E_{ST} = \int_\Sigma (p_{ij}^S u_i^T - p_{ij}^T u_i^S)\, dS_j.$$

We are now in a position to deal with the general case where internal stresses and inhomogeneities exist in the same body. The aggregate of inhomogeneities and sources of internal stress within a surface Σ are to be regarded as a complex singularity within it. The state of internal stress (so far as it does not arise from sources outside Σ) is to be defined in terms of the stress, strain or (where it can be defined) displacement in an infinite body with the same inhomogeneous elastic constants c_{ijkl} within Σ but homogeneous (with $c_{ijkl} = c_{ijkl}^0$) outside Σ. In the actual body we shall have as before $p_{ij}^S = p_{ij}^\infty + p_{ij}^I$, etc. The generalized image quantities thus defined arise from the presence of the boundary and also from the inhomogeneities lying between Σ and the surface of the body Σ_0. They can be found in principle as follows. Impose the strains e_{ij}^∞ on the body. The equations of equilibrium can only be satisfied if body forces $-(c_{ijkl}e_{kl}^\infty)_{,j}$ are present. Hence the image quantities are those which would be produced by body forces

$$f_i = (c_{ijkl}e_{kl}^\infty)_{,j}, \tag{37}$$

and surface tractions $-p_{ij}^\infty n_j$ acting on the body. The f_i are zero within Σ since $p_{ij,j}^\infty = 0$ there, the singularity being assumed as before to be in equilibrium in the infinite medium.

The force on the singularity in region II due to sources of stress in I and surface tractions will be

$$F_l = -(W_{\text{int.}} + W_{\text{ext.}})/\partial\xi_l, \tag{38}$$

where the internal stress and elastic constants depend on $\boldsymbol{\xi}$ in the following way:

$$\begin{aligned} &e_{ij}^\infty = e_{ij}^\infty(\mathbf{x}-\boldsymbol{\xi}) && \text{inside and outside } \Sigma,\\ &c_{ijkm} = c_{ijkm}(\mathbf{x}-\boldsymbol{\xi}) && \text{inside } \Sigma,\\ &c_{ijkm} \text{ is independent of } \boldsymbol{\xi} && \text{outside } \Sigma. \end{aligned}$$

For simplicity we shall assume that within a narrow shell containing Σ there are no sources of internal stress and $c_{ijkm} = c_{ijkm}^0$. This ensures that a displacement exists on Σ and that a small translation $\delta\xi_l$ does not engender discontinuities of c_{ijkm} across Σ.

In I we can write $e_{ij} = e'_{ij}(\mathbf{x}, \boldsymbol{\xi}^0) + e''_{ij}(\mathbf{x}, \boldsymbol{\xi})$ in terms of its value for some fixed value $\boldsymbol{\xi}^0$ of $\boldsymbol{\xi}$ and a variable part expressing the dependence on $\boldsymbol{\xi}$. Then although e'_{ij} cannot everywhere be derived from a displacement, e''_{ij} can. The contribution to (38) from the region I is

$$\begin{aligned} \frac{\partial W_{\text{int. II}}}{\partial\xi_l} &= \frac{1}{2}\int_{\text{I}} \frac{\partial}{\partial\xi_l}(p_{ij}e_{ij})\, dv = \int_{\text{I}} p_{ij}\frac{\partial e''_{ij}}{\partial\xi_l}\, dv \\ &= \int_{\text{I}} \left(p_{ij}\frac{\partial u''_i}{\partial\xi_l}\right)_{,j} dv = \int_{\Sigma_0-\Sigma} p_{ij}\frac{\partial u_i}{\partial\xi_l}\, dS_j, \end{aligned} \tag{39}$$

since $p_{ij,j} = 0$ and u'_i is independent of $\boldsymbol{\xi}$.

The contribution from the surface tractions is

$$\frac{\partial W_{\text{ext.}}}{\partial\xi_l} = -\int_{\Sigma_0} p_{ij}\frac{\partial u_i}{\partial\xi_l}\, dS_j. \tag{40}$$

In region II we put $e_{ij} = e_{ij}^{\infty} + e'_{ij}$. Then

$$\frac{\partial W_{\text{int. II}}}{\partial \xi_l} = \frac{1}{2}\frac{\partial}{\partial \xi_l}\int_{\text{II}} (p'_{ij}\,e'_{ij} + p_{ij}^{\infty}\,e_{ij}^{\infty} + p_{ij}^{\infty}\,e'_{ij} + p'_{ij}\,e_{ij}^{\infty})\,dv.$$

Since e'_{ij} can be derived from a displacement the first and third terms can be converted into surface integrals by writing, for example, $p'_{ij}\,e'_{ij}$ as $(p'_{ij}\,u'_i)_{,j}$ and using Gauss's theorem. So can the second, since although e_{ij}^{∞} cannot be derived from a displacement we have

$$\partial(p_{ij}^{\infty}\,e_{ij}^{\infty})/\partial \xi_l = -(p_{ij}^{\infty}\,e_{ij}^{\infty})_{,l}.$$

The fourth term is equal to the third. In this way we find

$$\frac{\partial W_{\text{int. II}}}{\partial \xi_l} = \frac{1}{2}\int_{\Sigma}\left\{\frac{\partial}{\partial \xi_l}\,(p'_{ij}\,u'_i + p_{ij}^{\infty}\,u'_i - p'_{ij}\,u_i^{\infty} - p_{ij}^{\infty}\,u_i^{\infty}) - (p_{ik}^{\infty}\,e_{ik}^{\infty})\,\delta_{lj}\right\}dS_j. \tag{41}$$

Adding (39), (40) and (41) we find after rearranging and using (11) that F_l differs from the right-hand side of (32) by the terms

$$\frac{1}{2}\int_{\Sigma}(p_{ij}^{\infty}\,e'_{ij} - p'_{ij}\,e_{ij}^{\infty})\,dS_l \tag{42}$$

and

$$\frac{1}{2}\int_{\Sigma}\left\{p'_{ij}\left(\frac{\partial}{\partial x_l} + \frac{\partial}{\partial \xi_l}\right)u'_i - u'_i\left(\frac{\partial}{\partial x_l} + \frac{\partial}{\partial \xi_l}\right)p'_{ij}\right\}dS_j. \tag{42$'$}$$

The expression (42) vanishes since $p_{ij}^{\infty}\,e'_{ij} = p'_{ij}\,e_{ij}^{\infty}$ on Σ. The divergence of the integrand of (42′) can be shown to vanish by expressing it entirely in terms of displacements and noting that $\partial(c_{ijkm})/\partial \xi_l = c_{ijkm,l}$ inside Σ. Consequently (42′) is zero. (The conditions for us to apply (8) and (9) to the terms in $\partial/\partial x_l$ and $\partial/\partial \xi_l$ separately are not satisfied.)

A more careful analysis shows that the requirement that c_{ijkm} shall be constant on Σ and in its neighbourhood is unnecessary. The choice of the constant c_{ijkm}^0 clearly makes no difference to the values of u_i^I, p_{ij}^I calculated from (37) nor to the values of u_i^{∞} and p_{ij}^I on Σ and hence cannot affect F_l. The modifications when body forces f_i are present in I are easily made. As externally applied forces they are to be treated on the same footing as the surface tractions. We thus have to add a term

$$-\int_{\text{I}} f_i\,\frac{\partial u''_i}{\partial \xi_l}\,dv$$

to (40). But since now $p_{ij} = -f_i$ an identical term must be subtracted from (41) and the rest of the calculation remains the same. If we wish to split F_l into F^A, F^I, etc., we have to show that (29) continues to hold even though the hypothetical infinite solid used to define p_{ij}^{∞} is inhomogeneous within Σ. It is easy to see that the stresses and displacements outside Σ are just those that would be produced by a system of body forces $-c_{ijkm}^0\,e_{km,j}$ within Σ if the interior of Σ were homogeneous, and (29) follows as before. Again, as noted in § 3, there is no reason why p_{ij}^{∞} should not be produced in whole or part by an equilibrating system of body forces $f_i^S = f_i^S(\mathbf{x} - \boldsymbol{\xi})$ within II. If they are present $p_{ij,j}^{\infty} = -f_i^S$ and a term

$$\frac{\partial}{\partial \xi_l}\int_{\text{II}} f_i^S\,u'_i\,dv$$

has to be added to the right-hand side of (41). F_l is then still given by the Maxwell tensor if $W_{\text{int. II}}$ is redefined as the sum of the elastic energy within II plus the potential energy of the sources of f_i^S.

It is thus clear that $F_l \delta\xi_l$ with

$$F_l = \int_\Sigma P_{jl}\, dS_j = \int_\Sigma (-p_{ij} u_{i,l} + \tfrac{1}{2} p_{ik} u_{i,k}\, \delta_{jl})\, dS_j$$

represents quite generally the decrease of the total potential energy of the system (namely, the elastic energy of the material and the potential energy of applied forces) when all sources of internal stress and inhomogeneity within Σ are given a small displacement $\delta\xi_l$. The energy $F_l \delta\xi_l$ is available for conversion into kinetic energy or dissipation by some process not considered in the elastic theory.

It will be seen from the foregoing that a singularity with a prescribed p_{ij}^∞ on and outside Σ can be regarded as arising from either (i) a suitable distribution of S_{ij} within Σ or (ii) a distribution of body forces f_i^S within Σ. The f_i^S are equal to the fictitious forces (36) associated with S_{ij}. F_l is the same for (i) and (ii). The representation (ii) fits into normal elastic theory where surface and body forces are regarded as the sources of the stress field. On the other hand (i) is a more natural representation of the singularities of physical theory, since they do not in fact arise from body forces.

The requirement of § 5 that surface tractions should be applied to a free surface of the body is rather unrealistic. It is certainly not satisfied, for example, when the body containing the singularity is a specimen in a tensile testing machine. We can, however, regard the specimen plus testing machine as a single body to which the results of this section may be applied. (Mechanical details can be accommodated to this picture, e.g. a bearing surface is a region where the shear modulus but not the bulk modulus vanishes.) Stresses are applied to the singularity by a weight rigidly connected to the machine (body forces), by tightening a screw (internal stress) or through a knife-edge (applied stresses according with the requirements of § 5).

It should be noted that there will be an image force (in the generalized sense) on a singularity even in a body of uniform composition if the orientation of c_{ijkm} varies with position, as it does on crossing a grain or twin boundary. There will thus be a force on a singularity near such a boundary quite apart from the effect of the array of dislocations into which the boundary can be decomposed.

Where there is a sudden discontinuity Δc_{ijkm} in the elastic constants across a surface Σ_d there will be a term $\Delta c_{ijkm}\, e_{kl}^\infty\, \partial H(\nu)/\partial x_j$ in the force (37) from which the image quantities are derived. Here ν is the distance measured from and normal to Σ_d and H is Heaviside's unit step function. Since $\partial H(\nu)/\partial x_j = \delta(\nu)\, \partial\nu/\partial x_j = \delta(\nu)\, n_j$, this term represents a surface distribution of force $\Delta c_{ijkm}\, e_{kl}^\infty n_j$ per unit area of Σ_d. When the c's vanish on one side of the surface we clearly get the surface traction $-p_{ij}^\infty n_j$ of § 4. In other words, we need not distinguish those image effects due to inhomogeneities from those due to the free surface; a finite body can be regarded as an infinite one for which the c's vanish outside Σ_0. Similarly, by letting the c's approach infinity on one side of Σ_d we can treat the image effect of a rigid wall, though the limiting process requires care.

Leibfried's (1949) expression for the interaction energy between a foreign atom and an edge dislocation does not agree with our result that the force between two singularities can be found by integrating P_{jl} over a surface separating them. Leibfried's model of the foreign atom is similar to Bilby's (cf. § 6), but the inner sphere is incompressible. According to our results the force on the foreign atom is given by the sum of (21) and (35) with p^A equal to the hydrostatic pressure produced by the dislocation, i.e. the force on a point centre of dilatation which produces the same stress field for $r > r_0$ plus a term quadratic in the dislocation stresses representing the force on the rigid sphere regarded as an inhomogeneity. With reversed sign the sum of these two terms is also the force which the foreign atom exerts on the dislocation, the inhomogeneity term becoming the force on the dislocation due to its image in the rigid sphere. If the interaction energy is to give the force correctly we must calculate it starting from an initial state independent of the relative position of the two singularities. A suitable initial state is furnished by the dislocation in an infinite homogeneous medium. First create a singularity of Bilby's type with its strength adjusted to equal Leibfried's. The work required for this is given by Leibfried's equation (18). Now cut out the centre of the Bilby singularity and insert the rigid sphere. The energy in the sphere cut out can be used to repay some of the energy expended. The interaction term in the energy of the sphere is found to be equal to Leibfried's (19 *b*). Hence the required interaction energy is the difference between Leibfried's (18) and (19 *b*), i.e. his equation (19 *a*). As an alternative starting point we may take the rigid inclusion with its stress field in the absence of the dislocation. Then, provided the elastic medium is attached to the surface of the inclusion, the interaction energy necessary to create the dislocation is again given (apart from image terms) by Leibfried's equation (19 *a*), in agreement with our results.

10. The dynamic case

The equations of motion of an elastic medium $p_{ij,j} - \rho\ddot{u}_i = 0$ (ρ is the density) are the Euler equations arising from a variational principle with the Lagrangian density function

$$\mathscr{L} = \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i - \tfrac{1}{2}c_{ijkl}u_{i,j}u_{k,l}$$

(Love 1927, p. 166). Associated with such a variational problem there is a 'canonical energy-momentum tensor'

$$T_{jl} = \frac{\partial\mathscr{L}}{\partial u_{i,j}}u_{i,l} - \mathscr{L}\delta_{jl} \quad (j, l = 1, 2, 3, 4;\ x_4 = t)$$

(cf. Wenzel 1949). This gives

$$T_{jl} = P_{jl} - \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i\delta_{jl} \quad (j, l = 1, 2, 3).$$

The components $T_{j4} = -p_{ij}\dot{u}_i$ form the vector giving the flux of energy (Love 1927, p. 177) and T_{44} is the energy density. The three components $T_{4l} = \rho\dot{u}_i u_{i,l}$ should represent a momentum density, but their interpretation is not immediate. It is natural to suggest that T_{jl} is the necessary generalization of P_{jl} for dynamical problems, though we can hardly hope to justify this by the simple kind of argument which served to find P_{jl}.

Linear elastic theory alone is insufficient to determine the motion of a singularity without some additional assumption. (Compare the problem of the equation of motion of a classical electron.) Consider, for example, a point singularity. We can write down a solution of the elastic equations which has a singular point moving in a prescribed manner. When the velocity of the singular point is not uniform there is a net radiation of energy from it. On the other hand, a singularity moving in an applied stress field absorbs energy from the latter in virtue of the force F_l^A. The physical counterpart of the elastic singularity has a finite energy density near the centre and cannot act as an unlimited source or sink of energy. A reasonable condition to impose on the elastic singularity is thus the following: the energy flux through a small sphere of radius r_0 about the singular point and moving with it is zero. This is sufficient to determine the equation of motion if the position of the singularity can be specified by a single parameter; in the general case we should have to try to interpret the momentum flux and apply a similar argument to it. For a centre of dilatation moving slowly in a straight line the motion is the same as that of a particle of mass $8\pi\rho\delta^2/r_0^3$ acted on by the force (21) (cf. also Burton 1892). It would, of course, be absurd to apply this result to the motion of an interstitial atom, whose motion is clearly not controlled merely by inertial forces. Frenkel & Kontorowa's (1938) caterpillar, in which it is the state of being interstitial rather than a particular atom which moves, is a closer analogue of a moving centre of dilatation, but even here dissipative effects are probably dominant.

A dislocation is a more hopeful example of an elastic singularity the motion of whose physical counterpart may perhaps be controlled chiefly by inertial forces. Consider a screw dislocation of infinite length parallel to the x_3-axis with its centre moving along the x_1-axis. The first step in finding its equation of motion is to calculate the displacement field surrounding it when it moves in a prescribed (supernatural) way. Suppose that the material is isotropic and that the dislocation is oscillating so that the position of its centre is $x_1 = \xi(t)$, $x_2 = 0$ with

$$\xi = l\,\mathrm{e}^{i\omega t}. \tag{43}$$

The displacement at the point of observation (x_1, x_2) at time t is

$$u_3 = -\tfrac{1}{4}ibklH_1^{(2)}(kr)\,\mathrm{e}^{i\omega t}\sin\theta \tag{43'}$$

plus a term independent of t (Eshelby 1949*a*), where

$$k = \omega/c, \quad c^2 = \mu/\rho, \quad r^2 = x_1^2 + x_2^2, \quad \tan\theta = x_2/x_1.$$

Apply to the left-hand sides of (43) and (43′) the operator (Lapwood 1949)

$$\frac{1}{2\pi i}\int_C (\ldots)\,\frac{d\omega}{\omega},$$

where C is the real ω-axis indented below the origin. Equation (43) becomes

$$\xi = lH(t), \tag{44}$$

representing a sudden jump of the dislocation through a distance l. The corresponding displacement, found by using the relations

$$\delta(t) = \frac{1}{2\pi}\int_C \mathrm{e}^{i\omega t}\,d\omega, \quad H_1^{(2)}(z) = -\frac{2}{\pi}\int_0^\infty \mathrm{e}^{-iz\cosh v}\cosh v\,dv$$

is

$$u_3 = -\frac{bl}{2\pi r}\sin\theta\,\frac{ct}{\sqrt{(c^2t^2 - r^2)}}\,H(ct - r).$$

A general motion $\xi = \xi(t)$ can be regarded as a succession of jumps of amount $l = \dot{\xi}(t)\,dt$ in time dt executed at the instantaneous position of the dislocation. Hence

$$u_3 = -\frac{b}{2\pi}x_2\int_{-\infty}^{\tau_0}\frac{c(t-\tau)}{[x_1 - \xi(\tau)]^2 + x_2^2}\,\frac{\dot{\xi}(\tau)\,d\tau}{\sqrt{\{c^2(t-\tau)^2 - [x_1 - \xi(\tau)]^2 - x_2^2\}}}, \tag{45}$$

where τ_0 is the value less than t for which the square root vanishes; it is unique if $\xi(\tau)$ is less than c for all τ less than t. Apart from its rather precarious derivation (45) has the required properties—it satisfies the elastic wave equation and has a discontinuity b across the line joining $(-\infty, 0)$ to the instantaneous position of the singular point.

It will be seen that the displacement and stress field depend on the whole previous motion. This is, of course, because following the jump (44) the effects arising from segments of the dislocation line with successively larger values of x_3 continue to arrive at a point in the plane $x_3 = 0$ ever afterwards. We may try to find the equation of motion by the energy argument used for the point singularity or by tapering the dislocation in the manner of figure 1 *c* and requiring that the external and self-forces should balance. Owing to the way in which the dislocation is haunted by its past we obtain an integral equation of motion, or equivalently a differential equation of infinite order.

The problem is even more intractable in the case of an arbitrary dislocation loop. The integral now extends only over a finite time interval but has a more involved spatial dependence and is complicated by the existence (in the isotropic case) of two velocities of wave propagation. Moreover, the problem is not simply one of translation of the loop; we have to find its change of form as a function of time. It is perhaps not worth while pursuing this problem until it is decided whether the motion of a dislocation is in fact governed by inertia (Frank 1949) or by quasi-viscous forces (Leibfried 1950).

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